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Variants of bosonization in parabosonic algebra: the Hopf and super-Hopf structures in parabosonic algebra

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Abstract
Parabosonic algebra in finite or infinite degrees of freedom is considered as a $\mathbb{Z}_2$-graded associative algebra, and is shown to be a $\mathbb{Z}_2$-graded (or super) Hopf algebra. The super-Hopf algebraic structure of the parabosonic algebra is established directly without appealing to its relation to the osp$(1/2n)$ Lie superalgebraic structure. The notion of super-Hopf algebra is equivalently described as a Hopf algebra in the braided monoidal category ${}_{\mathbb{C}\mathbb{Z}_2}\mathcal{M}$. The bosonization technique for switching a Hopf algebra in the braided monoidal category ${}_H\mathcal{M}$ (where H is a quasitriangular Hopf algebra) into an ordinary Hopf algebra is reviewed. In this paper, we prove that for the parabosonic algebra P_B, beyond the application of the bosonization technique to the original super-Hopf algebra, a bosonization-like construction is also achieved using two operators, related to the parabosonic total number operator. Both techniques switch the same super-Hopf algebra P_B to an ordinary Hopf algebra, thus producing two different variants of P_B, with an ordinary Hopf structure.

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1. Introduction

Parabosonic algebra has a long history both in theoretical and mathematical physics. Although formally introduced in the fifties by Green [10], Greenberg–Messiah [11] and Volkov [41] in the context of second quantization, its history traces back to the fundamental conceptual problems of quantum mechanics; in particular to Wigner's approach to first quantization [42]. In quantum mechanics, we consider a unital associative non-commutative algebra, generated in terms of the generators $p_i, q_i, I, i = 1, \ldots, n$ and relations (we have set $\hbar = 1$):

$$[q_i, p_j] = i\delta_{ij} I, \qquad [q_i, q_j] = [p_i, p_j] = 0 \tag{1}$$

where I is of course the unity of the algebra and $[x, y]$ stands for $xy - yx$. The states of the system are vectors of a Hilbert space, where the elements of the above-mentioned algebra act. The dynamics is determined by the Heisenberg equations of motion (we have set $\hbar = 1$):

$$\mathrm{i}\frac{\mathrm{d}q_i}{\mathrm{d}t} = [q_i, H], \qquad \mathrm{i}\frac{\mathrm{d}p_i}{\mathrm{d}t} = [p_i, H]. \tag{2}$$

Relations (1) are known in the physical literature as the Weyl algebra, or the Heisenberg–Weyl algebra or more commonly as the canonical commutation relations often abbreviated as CCR. Their central importance for the quantization procedure lies in the fact that if one accepts the algebraic relations (1) together with the quantum-dynamical equations (2) then it is an easy matter (see [6]) to extract the classical Hamiltonian equations of motion while on the other hand the acceptance of the classical Hamiltonian equations together with (1) reproduces the quantum dynamics exactly as described by (2). We do not consider arbitrary Hamiltonians of course but functions of the form $H = \sum_{i=1}^{n} p_i^2 + V(q_1, \ldots, q_n)$ which however are general enough for simple physical systems. In this way, the CCR emerge as a fundamental link between the classical and the quantum description of the dynamics.

For technical reasons, it is common to use—instead of the variables p_i, q_i—the linear combinations:

$$b_j^+ = \tfrac{1}{\sqrt{2}}(q_j - \mathrm{i}p_j), \qquad b_j^- = \tfrac{1}{\sqrt{2}}(q_j + \mathrm{i}p_j)$$

for $j = 1, \ldots, n$ in terms of which (1) become

$$\left[b_i^-, b_j^+\right] = \delta_{ij} I, \qquad \left[b_i^-, b_j^-\right] = \left[b_i^+, b_j^+\right] = 0 \tag{3}$$

for $i, j = 1, \ldots, n$. These latter relations are usually called the bosonic algebra (of n bosons), and in the case of the infinite degrees of freedom $i, j = 1, 2, \ldots$ they become the starting point of the free field theory (i.e., second quantization).

In 1950, E P Wigner in a two-page publication [42] noted that what the above approach implies is that the CCR (1) are sufficient conditions—but not necessary—for the equivalence between the classical Hamiltonian equations and the Heisenberg quantum-dynamical equations (2). In a kind of reversing the problem, Wigner posed the question of looking for necessary conditions for the simultaneous fulfillment of classical and quantum-dynamical equations. Working with the simplest example of a single, one-dimensional harmonic oscillator, he stated an infinite set of solutions for the above-mentioned problem. It is worth noting that a particular irreducible representation of the CCR was included as one special case among Wigner's infinite solutions.

A few years later in 1953, Green in his celebrated paper [10] introduced the parabosonic algebra (in possibly infinite degrees of freedom), by means of generators and relations:

$$\begin{aligned} &\left[B_m^-, \left\{B_k^+, B_l^-\right\}\right] = 2\delta_{km} B_l^- \\ &\left[B_m^-, \left\{B_k^-, B_l^-\right\}\right] = 0 \\ &\left[B_m^+, \left\{B_k^-, B_l^-\right\}\right] = -2\delta_{lm} B_k^- - 2\delta_{km} B_l^- \end{aligned} \tag{4}$$

$k, l, m = 1, 2, \ldots$ and $\{x, y\}$ stands for $xy + yx$. Green was primarily interested in field theoretic implications of the above-mentioned algebra, in the sense that he considered it as an alternative starting point for the second quantization problem, generalizing (3). However, despite his original motivation he was the first to realize—see also [28]—that Wigner's infinite solutions were nothing else but inequivalent irreducible representations of the parabosonic algebra (4). (See also the discussion in [30].)

This paper consists logically of two parts. The first part includes sections 2–4. The basic elements for the structure parabosonic algebra are presented. In section 2, we state the definition and derive basic properties of the parabosonic algebra in infinite degrees of freedom. The parabosonic algebra is considered to be a $\mathbb{Z}_2$-graded associative algebra with an infinite set of (odd) generators $B_i^{\pm}$ for $i = 1, 2, \ldots$. Its $\mathbb{Z}_2$-grading is inherited by the natural $\mathbb{Z}_2$-grading of the tensor algebra. The notions of $\mathbb{Z}_2$-graded algebra and $\mathbb{Z}_2$-graded tensor products [2] are discussed as a special examples of the more general and modern notions of $\mathbb{G}$-module algebras ($\mathbb{G}$: a finite Abelian group) and of braiding in monoidal categories [23, 24, 27]. In section 3, the notion of the super-Hopf algebra is presented in connection with the non-trivial quasitriangular structure of the $\mathbb{C}\mathbb{Z}_2$ group Hopf algebra and the braided monoidal category of its representations ${}_{\mathbb{C}\mathbb{Z}_2}\mathcal{M}$. The super-Hopf algebraic structure of the parabosonic algebra is established, without appealing to its Lie superalgebraic structure, and this is the central result of this part of the paper. Let us remark here in section 4, for the sake of completeness, well-known results regarding the Lie superalgebraic structure of the parabosonic algebra in finite degrees of freedom are reviewed.

The second part of the paper consists of section 5. We begin the section with a review of the bosonization technique for switching a Hopf algebra A in a braided monoidal category $\mathcal{C}$ into an ordinary Hopf algebra. Although we do not present the method in its full generality (see [22]), we give sufficient details for its application in a much more general class of problems than those involved in the 'super' or even in the $\mathbb{G}$-graded ($\mathbb{G}$ finite and Abelian) case: we consider the case of a Hopf algebra in the braided monoidal category ${}_H\mathcal{M}$ where H is some quasitriangular Hopf algebra, and explain in detail how we can construct an ordinary Hopf algebra out of it. The construction is achieved by means of a smash product algebra $A \star H$, and uses older results [26, 36], which guarantee the compatibility between the algebraic and the coalgebraic structure, in order for a smash product to be a Hopf algebra. The construction is such that the (braided) modules of the original (braided) Hopf algebra A and the (ordinary) modules of the 'bosonized' (ordinary) Hopf algebra $A \star H$ are in a bijective correspondence, providing thus an equivalence of categories. We apply the method in the case of the parabosonic algebra, i.e. the case for which $H = \mathbb{C}\mathbb{Z}_2$ equipped with its non-trivial quasitriangular structure, producing a 'variant' of the parabosonic algebra. This variant $P_B \star \mathbb{C}\mathbb{Z}_2$, which we will denote by $P_{B(g)}$, is a smash product Hopf algebra between the parabosonic super-Hopf algebra P_B and the group Hopf algebra $\mathbb{C}\mathbb{Z}_2$, and it is a Hopf algebra in the ordinary sense (and not in the 'super' sense). We explicitly state the structure maps (multiplication, comultiplication, counity and the antipode) for the (ordinary) Hopf algebraic structure of $P_{B(g)}$. Finally one more variant of the bosonization for the parabosonic algebra is presented, which achieves the same object with the bosonization technique. We construct an algebra $P_{B(K^{\pm})}$, which is little 'bigger' than the parabosonic algebra P_B or its bosonized form $P_{B(g)}$ and we establish its (ordinary) Hopf algebraic structure. So we prove that the bosonization technique is not unique.

In what follows, all vector spaces and algebras and all tensor products will be considered over the field of complex numbers. Whenever the symbol i enters a formula in another place than an index, it always denotes the imaginary unit $i^2 = -1$. Furthermore, whenever formulae from physics enter the text, we use the traditional convention: $\hbar = m = \omega = 1$. Finally, the Sweedler's notation for the comultiplication is freely used throughout the text.

2. Super-algebraic structure of Parabosons

The parabosonic algebra was originally defined in terms of generators and relations by Green [10] and Greenberg–Messiah [11]. We begin with restating their definition, in a modern algebraic context. Let us consider the vector space V_X freely generated by the elements: $X_i^+, X_j^-, i, j = 1, 2, \ldots$. Let $T(V_X)$ denote the tensor algebra of V_X:

$$T(V_X) = \bigoplus_{n\geqslant 0} T^n(V_X)$$

where $T^0(V_X) = \mathbb{C}, T^1(V_X) = V_X$ and for $n \geqslant 2$: $T^n(V_X) = V_X \otimes \cdots \otimes V_X$ the nth tensor power of V_X. It is well known [2] that $T(V_X)$ is—up to isomorphism—the free algebra generated by the elements X_i^+, X_j^- $(i, j = 1, 2, \ldots)$ of the basis of V_X or equivalently the non-commutative polynomial algebra generated over the indeterminates X_i^+, X_j^- $(i, j = 1, 2, \ldots)$. In $T(V_X)$, we consider the two-sided ideal I_{P_B}, generated by the following elements:

$$\left[\left\{X_i^{\xi}, X_j^{\eta}\right\}, X_k^{\epsilon}\right] - (\epsilon - \eta)\delta_{jk}X_i^{\xi} - (\epsilon - \xi)\delta_{ik}X_j^{\eta} \tag{5}$$

respectively, for all values of $\xi, \eta, \epsilon = \pm 1$ and $i, j = 1, 2, \ldots$. I_X is the unity of the tensor algebra. $[A, B]$ stands for $A\otimes B - B\otimes A$ and $\{A, B\}$ stands for $A\otimes B + B\otimes A$, where A and B are arbitrary elements of the tensor algebra $T(V_X)$. We now have the following definition.

Definition 2.1. *The parabosonic algebra in P_B is the quotient algebra of the tensor algebra $T(V_X)$ of V_X with the ideal I_{P_B}:*

$$P_B = T(V_X)/I_{P_B}.$$

We denote by $\pi_{P_B} : T(V_X) \to P_B$ the canonical projection. The elements X_i^+, X_j^-, I_X, where $i, j = 1, 2, \ldots$ and I_X is the unity of the tensor algebra, are the generators of the tensor algebra $T(V_X)$. The elements $\pi_{P_B}\left(X_i^+\right), \pi_{P_B}(X_j^-), \pi_{P_B}(I_X), i, j = 1, \ldots$ are a set of generators of the parabosonic algebra P_B, and they will be denoted by B_i^+, B_j^-, I for $i, j = 1, 2, \ldots$ respectively, from now on. $\pi_{P_B}(I_X) = I$ is the unity of the parabosonic algebra. The generators of the parabosonic algebra satisfy equation (4).

Based on the above definitions we prove now the following proposition.

Proposition 2.2. *The parabosonic algebra P_B is a $\mathbb{Z}_2$-graded associative algebra with its generators $B_i^{\pm}$ for $i, j = 1, 2, \ldots$, being odd elements.*

Proof. It is obvious that the tensor algebra $T(V_X)$ is a $\mathbb{Z}_2$-graded algebra with the monomials being homogeneous elements. If x is an arbitrary monomial of the tensor algebra, the degree of x is denoted by $|x| = \deg x$. Then $|x| = \deg(x) = 0$, namely x is an even element, if it constitutes of an even number of factors (an even number of generators of $T(V_X)$) and $|x| = \deg(x) = 1$, namely x is an odd element, if it constitutes of an odd number of factors (an odd number of generators of $T(V_X)$). The generators $X_i^+, X_j^-, i, j = 1, \ldots, n$ are odd elements in the above-mentioned gradation. In view of the above description, we can easily conclude that the $\mathbb{Z}_2$-gradation of the tensor algebra is immediately 'transferred' to the algebra P_B. The ideal I_{P_B} is an homogeneous ideal of the tensor algebra, since it is generated by homogeneous elements of $T(V_X)$. Consequently, the projection homomorphism π_{P_B} is an homogeneous algebra map of degree zero, or we can equivalently say that it is an even algebra homomorphism. □

The rise of the theory of quasitriangular Hopf algebras from the mid-1980s [5] and thereafter and especially the study and abstraction of their representations (see [23, 24, 27] and references therein) has provided us with a novel understanding of the notion and the properties of $\mathbb{G}$-graded algebras, where $\mathbb{G}$ is a finite Abelian group. We are restricting ourselves to the simplest case where $\mathbb{G} = \mathbb{Z}_2$ and we denote by $\{1, g\}$ the elements of the $\mathbb{Z}_2$ group (written multiplicatively). An algebra A being a $\mathbb{Z}_2$-graded algebra (in the physics literature the term superalgebra is also of widespread use) is equivalent to saying that A is a $\mathbb{C}\mathbb{Z}_2$-module algebra, via the $\mathbb{Z}_2$-action determined by

$$1 \triangleright a = a \qquad \text{and} \qquad g \triangleright a = (-1)^{|a|}a$$

for any a homogeneous in A and $|a|$ its degree.

What we actually mean is that A, apart from being an algebra is also a $\mathbb{CZ}_2$-module and at the same time the structure maps of A (i.e., the multiplication and the unity map which embeds the field into the center of the algebra) are $\mathbb{CZ}_2$-module maps, which is nothing else but homogeneous linear maps of degree 0 (or even linear maps). Stated more generally, the $\mathbb{G}$-grading of A can be equivalently described in terms of a specific action of the finite Abelian group $\mathbb{G}$ on A, thus in terms of a specific action of the $\mathbb{CG}$ group algebra on A. This is not something new. In fact such ideas already appear in works such as [3, 40].

In [24, 27], the construction of the tensor products of $\mathbb{G}$-graded objects is presented as a consequence of the quasitriangularity of the $\mathbb{CG}$ group Hopf algebra (for $\mathbb{G}$ a finite Abelian group, see [39]) or in other words: as a consequence of the braiding of the monoidal category ${}_{\mathbb{CG}}\mathcal{M}$ (category of $\mathbb{CG}$-modules).

It is well known that for any group $\mathbb{G}$, the group algebra $\mathbb{CG}$ equipped with the maps:

$$\Delta(z) = z \otimes z \qquad \varepsilon(z) = 1 \qquad S(z) = z^{-1}$$

for any $z \in \mathbb{G}$, becomes a Hopf algebra. Focusing again on the special case $\mathbb{G} = \mathbb{Z}_2$, the fact that A is a $\mathbb{Z}_2$-graded algebra is equivalently described by saying that A is an algebra in the braided monoidal category of $\mathbb{CZ}_2$-modules ${}_{\mathbb{CZ}_2}\mathcal{M}$. In this case the braiding is induced by the non-trivial quasitriangular structure of the $\mathbb{CZ}_2$ Hopf algebra i.e. by the non-trivial R-matrix:

$$R_{Z_2} = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g). \tag{6}$$

We digress here for a moment, to recall that (see [23, 24] or [27]) if (H, R_H) is a quasitriangular Hopf algebra through the R-matrix $R_H = \sum R_H^{(1)} \otimes R_H^{(2)}$, then the category of modules ${}_H\mathcal{M}$ is a braided monoidal category, where the braiding is given by a natural family of isomorphisms $\Psi_{V,W} : V \otimes W \cong W \otimes V$, given explicitly by

$$\Psi_{V,W}(v \otimes w) = \sum \left(R_H^{(2)} \triangleright w\right) \otimes \left(R_H^{(1)} \triangleright v\right) \tag{7}$$

for any $V, W \in obj({}_H\mathcal{M})$. By v, w we denote any elements of V, W respectively.

Combining equations (6) and (7) we immediately get the braiding in the ${}_{\mathbb{CZ}_2}\mathcal{M}$ category:

$$\Psi_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v. \tag{8}$$

This is a symmetric braiding, since

$$\Psi_{V,W} \circ \Psi_{W,V} = Id$$

so we actually have a symmetric monoidal category ${}_{\mathbb{CZ}_2}\mathcal{M}$, rather than a truly braided one.

The really important thing about the existence of the braiding (8) is that it provides us with an alternative way of forming tensor products of $\mathbb{Z}_2$-graded algebras. If A and B are superalgebras with multiplications:

$$m_A : A \otimes A \to A \qquad \text{and} \qquad m_B : B \otimes B \to B$$

respectively, then the super vector space $A \otimes B$ (with the obvious $\mathbb{Z}_2$-gradation) is equipped with the associative multiplication

$$(m_A \otimes m_B)(Id \otimes \Psi_{B,A} \otimes Id) : A \otimes B \otimes A \otimes B \longrightarrow A \otimes B \tag{9}$$

given equivalently by

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$$

for b, c homogeneous in B, A respectively. The tensor product becomes a superalgebra (or equivalently an algebra in the braided monoidal category of $\mathbb{CZ}_2$-modules ${}_{\mathbb{CZ}_2}\mathcal{M}$) which we will denote by $A \underline{\otimes} B$ and call the braided tensor product algebra from now on.

Let us close this description with two important remarks. First, we stress that in (9) both superalgebras A and B are viewed as $\mathbb{CZ}_2$-modules and as such we have $B \otimes A \cong A \otimes B$

through $b \otimes c \mapsto (-1)^{|c||b|} c \otimes b$. Second we underline that the tensor product (9) had been already known from the past [2] but rather as a special possibility of forming tensor products of superalgebras than as an example of the more general conceptual framework of the braiding applicable not only to superalgebras but to any $\mathbb{G}$-graded algebra ($\mathbb{G}$ a finite Abelian group) as long as $\mathbb{C}\mathbb{G}$ is equipped with a non-trivial quasitriangular structure or equivalently [27, 39], a bicharacter on $\mathbb{G}$ is given.

3. Super-Hopf structure of parabosons: a braided group

The notion of $\mathbb{G}$-graded Hopf algebra, for $\mathbb{G}$ a finite Abelian group, is not a new one neither in physics nor in mathematics. The idea appears already in the work of Milnor and Moore [25], where we actually have $\mathbb{Z}$-graded Hopf algebras. On the other hand, universal enveloping algebras of Lie superalgebras are widely used in physics and they are examples of $\mathbb{Z}_2$-graded Hopf algebras (see, e.g., [20, 38]). These structures are strongly resemblant of Hopf algebras but they are not Hopf algebras at least in the ordinary sense.

Restricting again to the simplest case where $\mathbb{G} = \mathbb{Z}_2$ we briefly recall this idea: an algebra A being a $\mathbb{Z}_2$-graded Hopf algebra (or super-Hopf algebra) means first of all that A is a $\mathbb{Z}_2$-graded associative algebra (or superalgebra). We now consider the braided tensor product algebra $A \underline{\otimes} A$. Then A is equipped with a coproduct

$$\underline{\Delta} : A \to A \underline{\otimes} A \tag{10}$$

which is an superalgebra homomorphism from A to the braided tensor product algebra $A \underline{\otimes} A$:

$$\underline{\Delta}(ab) = \sum (-1)^{|a_2||b_1|} a_1 b_1 \otimes a_2 b_2 = \underline{\Delta}(a) \cdot \underline{\Delta}(b)$$

for any a, b in A, with $\underline{\Delta}(a) = \sum a_1 \otimes a_2$, $\underline{\Delta}(b) = \sum b_1 \otimes b_2$, and a_2, b_1 homogeneous. We emphasize here that this is exactly the central point of difference between the 'super' and the 'ordinary' Hopf algebraic structure: in an ordinary Hopf algebra H we should have a coproduct $\Delta : H \to H \otimes H$ which should be an algebra homomorphism from H to the usual tensor product algebra $H \otimes H$.

Similarly, A is equipped with an antipode $\underline{S} : A \to A$ which is not an algebra anti-homomorphism (as it should be in an ordinary Hopf algebra) but a superalgebra anti-homomorphism (or 'twisted' anti-homomorphism, or braided anti-homomorphism) in the following sense (for any homogeneous $a, b \in A$):

$$\underline{S}(ab) = (-1)^{|a||b|} \underline{S}(b) \underline{S}(a). \tag{11}$$

The rest of the axioms which complete the super-Hopf algebraic structure (i.e., coassociativity, counity property and compatibility with the antipode) have the same formal description as in ordinary Hopf algebras.

Once again, the abstraction of the representation theory of quasitriangular Hopf algebras provides us with a language in which the above description becomes much more compact: we simply say that A is a Hopf algebra in the braided monoidal category of $\mathbb{C}\mathbb{Z}_2$-modules ${}_{\mathbb{C}\mathbb{Z}_2}\mathcal{M}$ or a braided group where the braiding is given in equation (8). What we actually mean is that A is simultaneously an algebra, a coalgebra and a $\mathbb{C}\mathbb{Z}_2$-module, while all the structure maps of A (multiplication, comultiplication, unity, counity and the antipode) are also $\mathbb{C}\mathbb{Z}_2$-module maps and at the same time the comultiplication $\underline{\Delta} : A \to A \underline{\otimes} A$ and the counit are algebra morphisms in the category ${}_{\mathbb{C}\mathbb{Z}_2}\mathcal{M}$ (see also [23, 24] or [27] for a more detailed description).

We proceed now to the proof of the following proposition which establishes the super-Hopf algebraic structure of the parabosonic algebra P_B.

Proposition 3.1. *The parabosonic algebra equipped with the even linear maps* $\underline{\Delta} : P_B \to P_B \underline{\otimes} P_B$, $\underline{S} : P_B \to P_B$, $\underline{\varepsilon} : P_B \to \mathbb{C}$, *determined by their values on the generators:*

$$\underline{\Delta}\big(B_i^{\pm}\big) = 1 \otimes B_i^{\pm} + B_i^{\pm} \otimes 1 \qquad \underline{\varepsilon}\big(B_i^{\pm}\big) = 0 \qquad \underline{S}\big(B_i^{\pm}\big) = -B_i^{\pm} \tag{12}$$

for $i = 1, 2, \ldots,$ *becomes a super-Hopf algebra.*

Proof. Recall that by definition $P_B = T(V_X)/I_{P_B}$. Consider the linear map:

$$\underline{\Delta}^T : V_X \to P_B \underline{\otimes} P_B$$

determined by its values on the basis elements specified by

$$\underline{\Delta}^T\big(X_i^{\pm}\big) = I \otimes B_i^{\pm} + B_i^{\pm} \otimes I.$$

By the universality of the tensor algebra this map is uniquely extended to a superalgebra homomorphism: $\underline{\Delta}^T : T(V_X) \to P_B \underline{\otimes} P_B$. After lengthy algebraic calculations (see the appendix A we can prove that

$$\underline{\Delta}^T\big(\big[\big\{X_i^{\xi}, X_j^{\eta}\big\}, X_k^{\epsilon}\big] - (\epsilon - \eta)\delta_{jk}X_i^{\xi} - (\epsilon - \xi)\delta_{ik}X_j^{\eta}\big) = 0 \tag{13}$$

for all values of $\xi, \eta, \epsilon = \pm 1$ and $i, j = 1, 2, \ldots.$ This means that $I_{P_B} \subseteq ker(\underline{\Delta}^T)$, which in turn implies that $\underline{\Delta}^T$ is uniquely extended to a superalgebra homomorphism: $\underline{\Delta} : P_B \to P_B \underline{\otimes} P_B$, according to the following (commutative) diagram:

$$\begin{tikzcd}
T(V_X) \arrow[rr, "\underline{\Delta}^T"] \arrow[dr, "\pi_{P_B}"'] & & P_B \underline{\otimes} P_B \\
& P_B \arrow[ur, dotted, "\underline{\Delta}"'] &
\end{tikzcd}$$

with values on the generators determined by (12).

Proceeding the same way we construct the maps $\underline{\varepsilon}$, $\underline{S}$, as determined in (12).

For the case of $\underline{\varepsilon}$, we start defining the trivial zero map

$$\underline{\varepsilon}^T : V_x \to \{0\} \in \mathbb{C}$$

and we (uniquely) extend its definition to a superalgebra homomorphism $\underline{\varepsilon} : P_B \to \mathbb{C}$ following the commutative diagram:

$$\begin{tikzcd}
T(V_X) \arrow[rr, "\underline{\varepsilon}^T"] \arrow[dr, "\pi_{P_B}"'] & & \mathbb{C} \\
& P_B \arrow[ur, dotted, "\underline{\varepsilon}"'] &
\end{tikzcd}$$

with values on the generators determined by (12).

In the case of the antipode $\underline{S}$ we need the notion of the $\mathbb{Z}_2$-graded opposite algebra (or opposite superalgera) P_B^{op}, which is a superalgebra defined as follows: P_B^{op} has the same underlying super vector space as P_B, but the multiplication is now defined as $a \cdot b = (-1)^{|a||b|}ba$, for all $a, b \in P_B$. (On the right-hand side, the product is of course the product of P_B.) We start by defining a linear map

$$\underline{S}^T : V_X \to P_B^{op}$$

determined by

$$\underline{S}^T\big(X_i^{\pm}\big) = -B_i^{\pm}.$$

This map is (uniquely) extended to a superalgebra homomorphism: $\underline{S}^T : T(V_X) \to P_B^{op}$. Now we can compute

$$\underline{S}^T\big(\big[\big\{X_i^{\xi}, X_j^{\eta}\big\}, X_k^{\epsilon}\big] - (\epsilon - \eta)\delta_{jk}X_i^{\xi} - (\epsilon - \xi)\delta_{ik}X_j^{\eta}\big) = 0 \tag{14}$$

for all values of $\xi, \eta, \epsilon = \pm 1$ and $i, j = 1, 2, \ldots$. This means that $I_{P_B} \subseteq ker(\underline{S}^T)$, which in turn implies that $\underline{S}^T$ is uniquely extended to a superalgebra homomorphism $\underline{S} : P_B \to P_B^{op}$, according to the following commutative diagram:

$$\begin{tikzcd} T(V_X) \arrow[rr, "\underline{S}^T"] \arrow[dr, "\pi_{P_B}"'] & & P_B^{op} \\ & P_B \arrow[ur, dotted, "\underline{S}"'] & \end{tikzcd}$$

thus to a superalgebra anti-homomorphism: $\underline{S} : P_B \to P_B$, with values on the generators determined by (12).

Now it is sufficient to verify the rest of the super-Hopf algebra axioms (coassociativity, counity and the compatibility condition for the antipode) on the generators of P_B. This can be done with straightforward computations. $\square$

Let us note here that the above proposition generalizes a result which—in the case of finite degrees of freedom—is a direct consequence of the work in [9]. In that work the parabosonic algebra in $2n$ generators (n-paraboson algebra) $P_B^{(n)}$ is shown to be isomorphic to the universal enveloping algebra of the orthosymplectic Lie superalgebra: $P_B^{(n)} \cong U(B(0, n))$. We present this accomplishment in detail in section 4. See also the discussion in [16].

4. Lie super-algebraic structure of parabosons: the case of finite degrees of freedom

In this section, we restrict ourselves to the case of the finite degrees of freedom (finite number of parabosons), in order to recall an important development in the study of the structure of the parabosonic algebra. We thus consider the parabosonic algebra generated by B_i^+, B_j^-, I, for $i, j = 1, 2, \ldots n$ where n is a positive integer. The generators satisfy exactly the same relations as before, determined by (4) or equivalently (5). The difference is that we only have a finite number of generators now and we will call this algebra the parabosonic algebra in $2n$ generators or the n-paraboson algebra from now on. We are going to denote it by $P_B^{(n)}$.

It was conjectured [29] that due to the mixing of commutators and anticommutators in $P_B^{(n)}$ the proper mathematical 'playground' for the study of the structure of $P_B^{(n)}$ should be some kind of Lie superalgebra ($\mathbb{Z}_2$-graded Lie algebra). Starting in the early 1980s, and using the recent (by that time) results in the classification of the finite-dimensional simple complex Lie superalgebras which was obtained by Kac (see [12, 13] but also [19]), Palev managed to identify the parabosonic algebra with the universal enveloping algebra of a certain simple complex Lie superalgebra. In [9, 31, 32], Palev shows the following lemma.

Lemma 4.1. *In the k-vector space $P_B^{(n)}$ we consider the k-subspace generated by the set of elements:*

$$\left\{\left\{B_i^\xi, B_j^\eta\right\}, B_k^\epsilon \middle| \xi, \eta, \epsilon = \pm, i, j, k = 1, \ldots, n\right\}.$$

This vector space is a superspace (i.e., a $\mathbb{Z}_2$-graded vector space or equivalently: a $\mathbb{C}\mathbb{Z}_2$-module).The elements B_i^ξ span the odd subspace and the elements $\left\{B_i^\xi, B_j^\eta\right\}$ span the even subspace.

The above vector space endowed with a bilinear multiplication $\langle .., .. \rangle$ whose values are determined by the values of the anticommutator and the commutator in $P_B^{(n)}$, i.e.,

$$\left\langle B_i^\xi, B_j^\eta \right\rangle = \left\{B_i^\xi, B_j^\eta\right\}$$

and

$$\left\langle \left\{B_i^\xi, B_j^\eta\right\}, B_k^\epsilon \right\rangle = \left[\left\{B_i^\xi, B_j^\eta\right\}, B_k^\epsilon\right] = (\epsilon - \eta)\delta_{jk} B_i^\xi + (\epsilon - \xi)\delta_{ik} B_j^\eta$$

respectively, according to the above-mentioned gradation, is a simple, complex Lie superalgebra (or $\mathbb{Z}_2$-graded Lie algebra) isomorphic to $B(0, n)$.

The elements

$$-\tfrac{1}{2}\{B_i^-, B_i^+\}, \qquad \{B_i^-, B_j^+\}, \qquad \{B_i^\xi, B_j^\xi\}, \qquad (B_i^\xi)^2, \qquad B_i^\xi$$

for all values $i \neq j = 1, 2, \ldots n$ and $\xi = \pm$, constitute a Cartan–Weyl basis of $B(0, n)$.

Note that, according to the above lemma, the elements $\{\{B_i^\xi, B_j^\eta\} | \xi, \eta = \pm, i, j = 1, \ldots, n\}$ constitute a basis in the even part of $B(0, n)$. This is a subalgebra of $B(0, n)$ isomorphic to the Lie algebra $sp(2n)$. Its Lie multiplication can be readily deduced from the above given commutators and reads

$$\langle\{B_i^\xi, B_j^\eta\}, \{B_k^\epsilon, B_l^\phi\}\rangle = [\{B_i^\xi, B_j^\eta\}, \{B_k^\epsilon, B_l^\phi\}] = (\epsilon - \eta)\delta_{jk}\{B_i^\xi, B_l^\phi\} + (\epsilon - \xi)\delta_{ik}\{B_j^\eta, B_l^\phi\} + (\phi - \eta)\delta_{jl}\{B_i^\xi, B_k^\epsilon\} + (\phi - \xi)\delta_{il}\{B_j^\eta, B_k^\epsilon\}.$$

On the other hand the elements $\{B_k^\epsilon | \epsilon = \pm, k = 1, \ldots, n\}$ constitute a basis of the odd part of $B(0, n)$.

Note also that $B(0, n)$ in Kac's notation is the classical simple complex orthosymplectic Lie superalgebra denoted as $\mathrm{osp}(1, 2n)$ in the notation traditionally used by physicists until then.

Based on the above observations, Palev finally proves the following proposition.

Proposition 4.2. *The parabosonic algebra in $2n$ generators is isomorphic to the universal enveloping algebra of the classical simple complex Lie superalgebra $B(0, n)$ (according to the classification of the simple complex Lie superalgebras given by Kac), i.e.,*

$$P_B^{(n)} \cong U(B(0, n)).$$

Lie superalgebras are exactly the algebraic structures underlying the idea of supersymmetry. The above-mentioned proposition thus indicates a link between parafield theories and supersymmetry. For a similar discussion one should also see [34].

Proposition 4.2 also indicates that in the case of the finite degrees of freedom, the representation theory of the parabosonic algebra $P_B^{(n)}$ coincides with the representation theory of the orthosymplectic Lie superalgebra $osp(1/2n)$ [14].

In the case of the finite degrees of freedom, the super-Hopf structure of the parabosonic algebra $P_B^{(n)}$ can be deduced from the fact that the universal enveloping algebra $U(L)$ of any Lie superalgebra L is an super-Hopf algebra. In the case of the infinite degrees of freedom, the parabosonic algebra is referred to the bibliography [30] to be also the universal enveloping algebra of some Lie superalgebra. Let us stress here, however, that our proof of proposition 3.1 does not make use of any kind of underlying Lie superalgebraic structure for either the $P_B^{(n)}$ or the P_B algebras.

5. Ordinary Hopf structures in parabosons

5.1. Review of the bosonization technique

A general scheme for transforming a Hopf algebra A in the braided monoidal category ${}_H\mathcal{M}$ (where H is a quasitriangular Hopf algebra) into an ordinary one, namely the smash product Hopf algebra $A \star H$, such that the category of braided modules of A and the category of (ordinary) modules of $A \star H$ are equivalent, has been developed in the original reference [22], see also [23, 24, 27]. The technique is called bosonization, the term coming from physics. This technique uses ideas developed by Molnar in [26] and by Radford in [36], which guarantee the compatibility between an algebraic and a coalgebraic structure in a tensor product [26] or even in a smash product [36], in order for it to become a bialgebra and finally a Hopf algebra. It is also presented and applied in [1, 7, 8]. For clarity reasons, we give a compact review the main points of the above method.

In general, A being a Hopf algebra in a category, means that A apart from being an algebra and a coalgebra, is also an object of the category and at the same time its structure maps (commultiplication, antipode etc) are morphisms in the category. In particular, if H is some quasitriangular Hopf algebra, A being a Hopf algebra in the braided monoidal category ${}_H\mathcal{M}$, means that the H-module A is an algebra in ${}_H\mathcal{M}$ (or H-module algebra) and a coalgebra in ${}_H\mathcal{M}$ (or H-module coalgebra) and at the same time Δ_A and ε_A are algebra morphisms in the category ${}_H\mathcal{M}$. (For more details on the above definitions one may consult for example [23, 24] or [27].)

Since A is an H-module algebra we can form the cross product algebra $A \rtimes H$ (also called: smash product algebra) which as a k-vector space is $A \otimes H$ (i.e., we write $a \rtimes h \equiv a \otimes h$ for every $a \in A, h \in H$), with multiplication given by

$$(b \otimes h)(c \otimes g) = \sum b(h_1 \rhd c) \otimes h_2 g \tag{15}$$

for all $b, c \in A$ and $h, g \in H$, the $\otimes$ the usual tensor product and $\Delta(h) = \sum h_1 \otimes h_2$.

On the other hand, A is a (left) H-module coalgebra with H, which is quasitriangular through the R-matrix $R_H = \sum R_H^{(1)} \otimes R_H^{(2)}$. Quasitriangularity switches the (left) action of H on A into a (left) coaction $\rho : A \to H \otimes A$ through

$$\rho(a) = \sum R_H^{(2)} \otimes \left(R_H^{(1)} \rhd a\right) \tag{16}$$

and A endowed with this coaction becomes (see [23, 24]) a (left) H-comodule coalgebra or equivalently a coalgebra in ${}^H\mathcal{M}$ (meaning that Δ_A and ε_A are (left) H-comodule morphisms, see [27]).

We recall here (see [23, 24]) that when H is a Hopf algebra and A is a (left) H-comodule coalgebra with the (left) H-coaction given by $\rho(a) = \sum a^{(1)} \otimes a^{(0)}$, one may form the cross coproduct coalgebra $A \rtimes H$, which as a k-vector space is $A \otimes H$ (i.e., we write $a \rtimes h \equiv a \otimes h$ for every $a \in A, h \in H$), with comultiplication given by

$$\Delta(a \otimes h) = \sum a_1 \otimes a_2^{(1)} h_1 \otimes a_2^{(0)} \otimes h_2 \tag{17}$$

and counit $\varepsilon(a \otimes h) = \varepsilon_A(a)\varepsilon_H(h)$. (In the above, $\Delta_A(a) = \sum a_1 \otimes a_2$ and we use in the elements of A upper indices included in parenthesis to denote the components of the coaction according to the Sweedler notation, with the convention that $a^{(i)} \in H$ for $i \neq 0$.)

Now we proceed by applying the above-described construction of the cross coproduct coalgebra $A \rtimes H$, with the special form of the (left) coaction given by equation (16). Replacing thus equation (16) into equation (17) we get for the special case of the quasitriangular Hopf algebra H the cross coproduct comultiplication:

$$\Delta(a \otimes h) = \sum a_1 \otimes R_H^{(2)} h_1 \otimes \left(R_H^{(1)} \rhd a_2\right) \otimes h_2. \tag{18}$$

Finally we can show that the cross product algebra (with multiplication given by (15)) and the cross coproduct coalgebra (with comultiplication given by (18)) fit together and form a bialgebra (see [23, 24, 26, 27, 36]). This bialgebra, furnished with an antipode

$$S(a \otimes h) = (S_H(h_2))u(R^{(1)} \rhd S_A(a)) \otimes S(R^{(2)} h_1) \tag{19}$$

where $u = \sum S_H(R^{(2)})R^{(1)}$, and S_A the (braided) antipode of A, becomes (see [23]) an ordinary Hopf algebra. This is the smash product Hopf algebra denoted by $A \star H$.

Apart from the above-described construction, it is worth mentioning two more important points proved in [22]. First, it is shown that if H is triangular and A is quasitriangular in the category ${}_H\mathcal{M}$, then $A \star H$ is (ordinarily) quasitriangular. Second, it is shown that the braided modules of the original braided Hopf algebra A (A-modules in ${}_H\mathcal{M}$, where A is an algebra in ${}_H\mathcal{M}$) and the (ordinary) modules of the 'bosonized' (ordinary) Hopf algebra $A \star H$ are in a bijective correspondence, providing thus an equivalence of categories. The category of the braided modules of A (A-modules in ${}_H\mathcal{M}$) where the braiding is given by a natural family of isomorphisms $\Psi_{V,W} : V \otimes W \cong W \otimes V$, stated explicitly by

$$\Psi_{V,W}(v \otimes w) = \sum \left(R_H^{(2)} \triangleright w\right) \otimes \left(R_H^{(1)} \triangleright v\right) \tag{20}$$

for any $V, W \in obj({}_H\mathcal{M})$ (by v, w we denote any elements of V, W respectively), is equivalent to the category of the (ordinary) modules of $A \star H$. Let us stress here that from the mathematicians viewpoint, this does not prove that we have a Morita equivalence, since such a kind of equivalence would presuppose the whole category of modules over A and not its subcategory of braided modules.

Let us close this review of the bosonization technique, with a note on terminology. The term 'bosonization' was first introduced by Majid in [22]. It is coming from physics and it stems from the—widespread among physicists—point of view which considers the bosonic algebra to be a quotient algebra of the universal enveloping algebra of the Heisenberg Lie algebra, with its elements thus being even or: ungraded elements.

In the case that $H = \mathbb{C}\mathbb{G}$ where $\mathbb{G}$ is a finite Abelian group, the Hopf algebra in ${}_{\mathbb{C}\mathbb{G}}\mathcal{M}$ is just a $\mathbb{G}$-graded Hopf algebra in the sense of [20, 27] or [38]. The result of the bosonization technique in this case is the construction of an ordinary Hopf algebra $A \star \mathbb{C}\mathbb{G}$ which absorbs the grading and whose elements are ungraded or 'bosonic' elements. This is the original motivation which led Majid to the use of the term bosonization (see also [23, 24]).

Finally, let us note that for another use of the term bosonization, which is technically reminiscent of the above but it is not explicitly related to the Hopf structure, one should also see [35].

5.2. *Bosonization of P_B using the smash product*

In the special case that A is some super-Hopf algebra, then $H = \mathbb{C}\mathbb{Z}_2$, equipped with its non-trivial quasitriangular structure, formerly mentioned. In this case, the technique simplifies and the ordinary Hopf algebra produced is the smash product Hopf algebra $A \star \mathbb{C}\mathbb{Z}_2$. The grading in A is induced by the $\mathbb{C}\mathbb{Z}_2$-action on A:

$$1 \triangleright a = a, \qquad g \triangleright a = (-1)^{|a|}a \tag{21}$$

for a homogeneous in A. Utilizing the non-trivial R-matrix R_g and using equations (6) and (16) we can readily deduce the form of the induced $\mathbb{C}\mathbb{Z}_2$-coaction on A:

$$\rho(a) = g^{|a|} \otimes a \equiv \begin{cases} 1 \otimes a, & a : \text{even} \\ g \otimes a, & a : \text{odd}. \end{cases} \tag{22}$$

Let us note here that instead of invoking the non-trivial quasitriangular structure R_g we could alternatively extract the (left) coaction (22) utilizing the self-duality of the $\mathbb{C}\mathbb{Z}_2$ Hopf algebra. For any Abelian group $\mathbb{G}$ a (left) action of $\mathbb{C}\mathbb{G}$ coincides with a (right) action of $\mathbb{C}\mathbb{G}$. On the other hand, for any finite group, a (right) action of $\mathbb{C}\mathbb{G}$ is the same thing as a (left) coaction of the dual Hopf algebra $(\mathbb{C}\mathbb{G})^*$. Since $\mathbb{C}\mathbb{Z}_2$ is both finite and Abelian and hence self-dual in the sense that $\mathbb{C}\mathbb{Z}_2 \cong (\mathbb{C}\mathbb{Z}_2)^*$ as Hopf algebras, it is immediate to see that the (left) action (21) and the (left) coaction (22) are virtually the same thing.

The above-mentioned action and coaction enable us to form the cross product algebra and the cross coproduct coalgebra according to the preceding discussion which finally form the smash product Hopf algebra $A \star \mathbb{C}\mathbb{Z}_2$. The grading of A, is 'absorbed' in $A \star \mathbb{C}\mathbb{Z}_2$, and becomes an inner automorphism:

$$gag = (-1)^{|a|}a$$

where we have identified $a \star 1 \equiv a$ and $1 \star g \equiv g$ in $A \star \mathbb{C}\mathbb{Z}_2$, and a is the homogeneous element in A. This inner automorphism is exactly the adjoint action of g on $A \star \mathbb{C}\mathbb{Z}_2$ (as an ordinary Hopf algebra). The following proposition is proved—as an example of the bosonization technique—in [23]:

Proposition 5.1. *Corresponding to every super-Hopf algebra A there is an ordinary Hopf algebra $A \star \mathbb{C}\mathbb{Z}_2$, its bosonization, consisting of A extended by adjoining an element g with relations, coproduct, counit and antipode:*

$$\begin{array}{llll} g^2 = 1 & ga = (-1)^{|a|}ag & \Delta(g) = g \otimes g & \Delta(a) = \sum a_1 g^{|a_2|} \otimes a_2 \\ S(g) = g & S(a) = g^{-|a|}\underline{S}(a) & \varepsilon(g) = 1 & \varepsilon(a) = \underline{\varepsilon}(a) \end{array} \tag{23}$$

where $\underline{S}$ and $\underline{\varepsilon}$ denote the original maps of the super-Hopf algebra A.

In the case that A is super-quasitriangular via the R-matrix

$$\underline{R} = \sum \underline{R}^{(1)} \otimes \underline{R}^{(2)}$$

then the bosonized Hopf algebra $A \star \mathbb{C}\mathbb{Z}_2$ is quasitriangular (in the ordinary sense) via the R-matrix:

$$R_{smash} = R_{Z_2} \sum \underline{R}^{(1)} g^{|\underline{R}^{(2)}|} \otimes \underline{R}^{(2)}.$$

Moreover, the representations of the bosonized Hopf algebra $A \star \mathbb{C}\mathbb{Z}_2$ are precisely the super-representations of the original superalgebra A.

The application of the above proposition in the case of the parabosonic algebra P_B is straightforward, we immediately get its bosonized form $P_{B(g)}$ which by definition is $P_{B(g)} \equiv P_B \star \mathbb{C}\mathbb{Z}_2$. Utilizing equations (12) which describe the super-Hopf algebraic structure of the parabosonic algebra P_B, and replacing them into equations (23) which describe the ordinary Hopf algebra structure of the bosonized superalgebra, we get after straightforward calculations the explicit form of the (ordinary) Hopf algebra structure of $P_{B(g)} \equiv P_B \star \mathbb{C}\mathbb{Z}_2$ which reads

$$\begin{array}{llll} \Delta\left(B_i^{\pm}\right) = B_i^{\pm} \otimes 1 + g \otimes B_i^{\pm} & \Delta(g) = g \otimes g & \varepsilon\left(B_i^{\pm}\right) = 0 & \varepsilon(g) = 1 \\ S\left(B_i^{\pm}\right) = B_i^{\pm} g = -g B_i^{\pm} & S(g) = g & g^2 = 1 & \left\{g, B_i^{\pm}\right\} = 0 \end{array} \tag{24}$$

where $i = 1, 2, \ldots$ and we have again identified $B_i^{\pm} \star 1 \equiv B_i^{\pm}$ and $1 \star g \equiv g$ in $P_B \star \mathbb{C}\mathbb{Z}_2$.

Finally, we can easily check that since $\mathbb{C}\mathbb{Z}_2$ is triangular (via R_{Z_2}) and P_B is super-quasitriangular (trivially since it is super-cocommutative) it is an immediate consequence of the above proposition that $P_{B(g)}$ is quasitriangular (in the ordinary sense) via the R-matrix:

$$R_{\text{smash}} = \tfrac{1}{2}(1 \star 1 \otimes 1 \star 1 + 1 \star 1 \otimes 1 \star g + 1 \star g \otimes 1 \star 1 - 1 \star g \otimes 1 \star g) \tag{25}$$

which under the above-mentioned identification: $1 \star g \equiv g$ completely coincides with the R-matrix R_{Z_2} given in equation (6).

5.3. Bosonization of P_B using two additional operators $K^{\pm}$

Let us describe now a different construction (see also [4, 16] for the case of the finite degrees of freedom and [15] for the general case), which achieves the same object, i.e. the determination of an ordinary Hopf structure for the parabosonic algebra P_B.

Proposition 5.2. *Corresponding to the super-Hopf algebra P_B there is an ordinary Hopf algebra $P_{B(K^{\pm})}$, consisting of P_B extended by adjoining two elements K^+, K^- with relations, coproduct, counit and antipode:*

$$\begin{array}{ll} \Delta\big(B_i^{\pm}\big) = B_i^{\pm} \otimes 1 + K^{\pm} \otimes B_i^{\pm} & \Delta(K^{\pm}) = K^{\pm} \otimes K^{\pm} \\ \varepsilon\big(B_i^{\pm}\big) = 0 & \varepsilon(K^{\pm}) = 1 \\ S\big(B_i^{\pm}\big) = B_i^{\pm} K^{\mp} & S(K^{\pm}) = K^{\mp} \\ K^+K^- = K^-K^+ = 1 & \big\{K^+, B_i^{\pm}\big\} = 0 = \big\{K^-, B_i^{\pm}\big\} \end{array} \tag{26}$$

for all values $i = 1, 2, \ldots.$

Proof. Consider the complex vector space $\mathbb{C}\langle X_i^+, X_j^-, K^{\pm}\rangle$ freely generated by the elements X_i^+, X_j^-, K^+, K^- where $i = 1, 2, \ldots$. Denote $T\big(X_i^+, X_j^-, K^{\pm}\big)$ its tensor algebra. In the tensor algebra we denote I_{BK} the ideal generated by all the elements of the form (5) together with all elements of the form: $K^+K^- - 1$, $K^-K^+ - 1$, $\big\{K^+, X_i^{\pm}\big\}$, $\big\{K^-, X_i^{\pm}\big\}$, for all values of $i = 1, 2, \ldots$. We define

$$P_{B(K^{\pm})} = T\big(X_i^+, X_j^-, K^{\pm}\big)\big/ I_{BK}.$$

We denote by $B_i^{\pm}, K^{\pm}$ where $i = 1, 2, \ldots$ the images of the generators $X_i^{\pm}, K^{\pm}, i = 1, 2, \ldots$ of the tensor algebra, under the canonical projection. These are a set of generators of $P_{B(K^{\pm})}$.

Consider the linear map

$$\Delta^T : \mathbb{C}\langle X_i^+, X_j^-, K^{\pm}\rangle \to P_{B(K^{\pm})} \otimes P_{B(K^{\pm})}$$

determined by

$$\begin{aligned} \Delta^T\big(X_i^{\pm}\big) &= B_i^{\pm} \otimes 1 + K^{\pm} \otimes B_i^{\pm} \\ \Delta^T(K^{\pm}) &= K^{\pm} \otimes K^{\pm}. \end{aligned}$$

By the universality property of the tensor algebra, this map is uniquely extended to an algebra homomorphism:

$$\Delta^T : T\big(X_i^+, X_j^-, K^{\pm}\big) \to P_{B(K^{\pm})} \otimes P_{B(K^{\pm})}.$$

We emphasize that the usual tensor product algebra $P_{B(K^{\pm})} \otimes P_{B(K^{\pm})}$ is now considered, with multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$ for any $a, b, c, d \in P_{B(K^{\pm})}$. Now we can trivially verify that

$$\Delta^T\big(\big\{K^{\pm}, X_i^{\pm}\big\}\big) = \Delta^T(K^+K^- - 1) = \Delta^T(K^-K^+ - 1) = 0. \tag{27}$$

After lengthy algebraic calculations we also get

$$\Delta^T\big(\big[\big\{X_i^{\xi}, X_j^{\eta}\big\}, X_k^{\epsilon}\big] - (\epsilon - \eta)\delta_{jk} X_i^{\xi} - (\epsilon - \xi)\delta_{ik} X_j^{\eta}\big) = 0. \tag{28}$$

The calculations are carried in the same spirit of the calculation found in the appendix A but we note that this time we use the comultiplication stated in equation (26) and the usual tensor product algebra $P_{B(K^{\pm})} \otimes P_{B(K^{\pm})}$ is considered instead of the braided tensor product algebra $P_{B(K^{\pm})} \underline{\otimes} P_{B(K^{\pm})}$ used in Appendix appendix A.

Relations (27) and (28) mean that $I_{BK} \subseteq ker(\Delta^T)$ which in turn implies that Δ^T is uniquely extended to an algebra homomorphism from $P_{B(K^{\pm})}$ to the usual tensor product algebra $P_{B(K^{\pm})} \otimes P_{B(K^{\pm})}$, with the values on the generators determined by (26), according to the following (commutative) diagram:

$$\begin{tikzcd}
T(X_i^+, X_j^-, K^{\pm}) \arrow[rr, "\Delta^T"] \arrow[dr, "\pi"'] & & P_{B(K^{\pm})} \otimes P_{B(K^{\pm})} \\
& P_{B(K^{\pm})} \arrow[ur, dotted, "\Delta"] &
\end{tikzcd}$$

Following the same procedure, we construct an algebra homomorphism $\varepsilon : P_{B(K^{\pm})} \rightarrow \mathbb{C}$ and an algebra antihomomorphism $S : P_{B(K^{\pm})} \rightarrow P_{B(K^{\pm})}$ which are completely determined by their values on the generators of $P_{B(K^{\pm})}$ as given in (26). Note that in the case of the antipode we start by defining a linear map S^T from $\mathbb{C}\langle X_i^+, X_j^-, K^{\pm}\rangle$ to the opposite algebra $P^{op}_{B(K^{\pm})}$, with values determined by $S^T\left(X_i^{\pm}\right) = B_i^{\pm}K^{\mp}$ and $S^T(K^{\pm}) = K^{\mp}$. Following the above-described procedure, we verify that $I_{BK} \subseteq ker(S^T)$, thus resulting with an algebra anti-homomorphism:

$$S : P_{B(K^{\pm})} \rightarrow P_{B(K^{\pm})}$$

with values on the generators determined by (26).

Now it is sufficient to verify the rest of the Hopf algebra axioms (i.e., coassociativity of Δ, counity property for ε, and the compatibility condition which ensures us that S is an antipode) on the generators of $P_{B(K^{\pm})}$. This can be done with straightforward computations (see [4]). □

Let us note here that the initiation for the above-mentioned construction lies in the case of the finite degrees of freedom: if we consider the parabosonic algebra in $2n$ generators (n-paraboson algebra) and denote it by $P_B^{(n)}$, it is possible to construct explicit realizations of the elements K^+ and K^- in terms of formal power series, such that the relations specified in (26) hold. The construction is briefly (see also [4]) as follows: we define

$$\mathcal{N} = \sum_{i=1}^{n} N_{ii} = \frac{1}{2}\sum_{i=1}^{n}\left\{B_i^+, B_i^-\right\}$$

We inductively prove

$$\mathcal{N}^m B_i^{\pm} = B_i^{\pm}(\mathcal{N} \pm I)^m .$$

For any entire complex function $f(z)$ we get

$$f(\mathcal{N})\, B_i^{\pm} = B_i^{\pm} f(\mathcal{N} + I) .$$

We now introduce the following elements:

$$K^+ = \exp(\mathrm{i}\pi\mathcal{N}), \qquad K^- = \exp(-\mathrm{i}\pi\mathcal{N})$$

then we get

$$\left\{K^+, B_i^{\pm}\right\} = 0, \qquad \left\{K^-, B_i^{\pm}\right\} = 0. \tag{29}$$

A direct application of the Baker–Campbell–Hausdorff formula leads also to

$$K^+K^- = K^-K^+ = 1 \tag{30}$$

which completes the statement.

6. Discussion

Several points which deserve to be discussed stem from the constructions of the preceding paragraphs.

First of all we should mention that an analogous treatment regarding the (super-) algebraic and the (super-) Hopf algebraic structure can be given for the parafermionic algebras and for mixed systems of paraparticles as well. The parafermionic algebra in finite degrees of freedom has been shown [18, 37], to be isomorphic to the universal enveloping algebra of the Lie algebra $B_n = so(2n+1)$ and thus an ordinary Hopf algebra [4], consequently the grading does not seem to play an important role in its structure. On the other hand, algebras which describe mixed systems of paraparticles such as the relative parabose or the relative parafermi sets (see [11] for their description) have been shown to be $\mathbb{Z}_2$-graded (see [33]) or $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded (see [43]) algebras respectively. It would thus be an interesting idea to apply similar techniques to these algebras and obtain results about their braided representations and their tensor products, and about their super-Hopf structure and their bosonized forms as well. Of course such questions inevitably involve questions of pure mathematical interest, such as the possible quasitriangular structures (and thus the possible braidings) for a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra, which up to our knowledge have not yet been solved in general (see [39] for a relevant discussion).

Let us note here that the super-Hopf algebraic structure of the parabosonic algebra established in section 3 has an application: it has recently been shown [17] that using the results of proposition 3.1, one may obtain the construction of the parabosonic Fock-like representations corresponding to an arbitrary value of the positive integer p (see [11]) as irreducible submodules of the braided tensor product representations between p-copies of the first Fock-like representation (corresponding to the value of $p = 1$). The super-Hopf algebraic structure of the parabosonic algebra is essential in this process and leads us to a purely braided interpretation of the Green ansatz for parabosons (see [17] for a more detailed description of the method).

Regarding now the results of the last section, i.e., the 'bosonized' variants $P_{B(g)}$, $P_{B(K^{\pm})}$ of the parabosonic algebras, various questions can be posed.

From the point of view of the structure, an obvious question arises: while $P_{B(g)}$ is a quasitriangular Hopf algebra through the R-matrix, R_{Z_2} given in equation (6), there is yet no suitable R-matrix for the Hopf algebra $P_{B(K^{\pm})}$. Thus the question of the quasitriangular structure of $P_{B(K^{\pm})}$ is open.

On the other hand, regarding representations, we have already noted that the super representations of P_B ($\mathbb{Z}_2$-graded modules of P_B or equivalently: P_B-modules in ${}_{\mathbb{C}\mathbb{Z}_2}\mathcal{M}$) are in '1–1' correspondence with the (ordinary) representations of $P_{B(g)}$. The construction of the representation of $P_{B(g)}$ which corresponds to any given representation of P_B can be done straightforwardly [22, 23]. Although we do not have such a strong result for the representations of $P_{B(K^{\pm})}$, the construction in the end of section 5 for the case of finite degrees of freedom, enables us to uniquely extend the Fock-like [11] representations of $P_B^{(n)}$ to representations of $P_{B(K^{\pm})}^{(n)}$. Since the Fock-like representations of P_B are unique up to unitary equivalence (see the proof in [11] or [28]), this is a point which deserves to be discussed analytically in a forthcoming work. We must note here that this question has to be discussed in connection with the explicit construction of the parabosonic Fock-like representations which is yet another open problem (see the discussion in [17] or [21]).

Finally, it will be an interesting thing to study the (ordinary) tensor products of representations of $P_{B(g)}$ and $P_{B(K^{\pm})}$, through the comultiplications stated in (24) and (26) respectively, in comparison with the (braided) tensor products of (braided) representations of P_B through the comultiplication stated in (12). Specifically, it will be of interest to answer the question of whether the ordinary Hopf structures presented in the last section of this paper are capable of generating essentially new representations of the parabosonic algebra: the possibility that the reduction of (ordinary) tensor product representations of either $P_{B(g)}$ or $P_{B(K^{\pm})}$ might lead to submodules non-equivalent to the parabosonic Fock-like representations (the latter emerge as irreducible submodules in the reduction of the braided tensor product representations of P_B) is an intriguing one and deserves to be discussed analytically in a forthcoming work.

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Appendix. Proof of equation (13)

Using the fact that the generators of the parabosonic algebra P_B are odd elements and the multiplication in the braided tensor product algebra $P_B \underline{\otimes} P_B$ is given by (9), we have $\left(I \otimes B_i^{\xi}\right)\left(B_j^{\eta} \otimes I\right) = -B_j^{\eta} \otimes B_i^{\xi}$ while $\left(B_j^{\eta} \otimes I\right)\left(I \otimes B_i^{\xi}\right) = B_j^{\eta} \otimes B_i^{\xi}$ in $P_B \underline{\otimes} P_B$. Now we compute

$$\begin{aligned}
\underline{\Delta}^T\left(\left\{X_i^{\xi}, X_j^{\eta}\right\}\right) &= \underline{\Delta}^T\left(X_i^{\xi}X_j^{\eta} + X_j^{\eta}X_i^{\xi}\right) = \underline{\Delta}^T\left(X_i^{\xi}\right)\underline{\Delta}^T\left(X_j^{\eta}\right) + \underline{\Delta}^T\left(X_j^{\eta}\right)\underline{\Delta}^T\left(X_i^{\xi}\right) \\
&= \left(B_i^{\xi} \otimes I + I \otimes B_i^{\xi}\right)\left(B_j^{\eta} \otimes I + I \otimes B_j^{\eta}\right) + \left(B_j^{\eta} \otimes I + I \otimes B_j^{\eta}\right)\left(B_i^{\xi} \otimes I + I \otimes B_i^{\xi}\right) \\
&= B_i^{\xi}B_j^{\eta} \otimes I + B_i^{\xi} \otimes B_j^{\eta} - B_j^{\eta} \otimes B_i^{\xi} + I \otimes B_i^{\xi}B_j^{\eta} + +B_j^{\eta}B_i^{\xi} \otimes I \\
&\quad + B_j^{\eta} \otimes B_i^{\xi} - B_i^{\xi} \otimes B_j^{\eta} + I \otimes B_j^{\eta}B_i^{\xi} = I \otimes \left\{B_i^{\xi}, B_j^{\eta}\right\} + \left\{B_i^{\xi}, B_j^{\eta}\right\} \otimes I.
\end{aligned}$$

So we have proved that for the even elements $\left\{X_i^{\xi}, X_j^{\eta}\right\}$ (for all values of $\xi, \eta, = \pm 1$ and $i, j = 1, 2, \ldots$) of the tensor algebra $T(V_X)$ we have

$$\underline{\Delta}^T\left(\left\{X_i^{\xi}, X_j^{\eta}\right\}\right) = \left\{B_i^{\xi}, B_j^{\eta}\right\} \otimes I + I \otimes \left\{B_i^{\xi}, B_j^{\eta}\right\} \tag{A.1}$$

Using result (A.1) and the fact that $\left\{X_i^{\xi}, X_j^{\eta}\right\}$ (for all values of $\xi, \eta, = \pm 1$ and $i, j = 1, 2, \ldots$) are even elements, we get

$$\begin{aligned}
\underline{\Delta}^T\left(\left[\left\{X_i^{\xi}, X_j^{\eta}\right\}, X_k^{\epsilon}\right]\right) &= \underline{\Delta}^T\left(\left\{X_i^{\xi}, X_j^{\eta}\right\}\right)\underline{\Delta}^T\left(X_k^{\epsilon}\right) - \underline{\Delta}^T\left(X_k^{\epsilon}\right)\underline{\Delta}^T\left(\left\{X_i^{\xi}, X_j^{\eta}\right\}\right) \\
&= \left(\left\{B_i^{\xi}, B_j^{\eta}\right\} \otimes I + I \otimes \left\{B_i^{\xi}, B_j^{\eta}\right\}\right)\left(B_k^{\epsilon} \otimes I + I \otimes B_k^{\epsilon}\right) \\
&\quad - \left(B_k^{\epsilon} \otimes I + I \otimes B_k^{\epsilon}\right)\left(\left\{B_i^{\xi}, B_j^{\eta}\right\} \otimes I + I \otimes \left\{B_i^{\xi}, B_j^{\eta}\right\}\right) \\
&= \left\{B_i^{\xi}, B_j^{\eta}\right\}B_k^{\epsilon} \otimes I + \left\{B_i^{\xi}, B_j^{\eta}\right\} \otimes B_k^{\epsilon} + B_k^{\epsilon} \otimes \left\{B_i^{\xi}, B_j^{\eta}\right\} + I \otimes \left\{B_i^{\xi}, B_j^{\eta}\right\}B_k^{\epsilon} \\
&\quad - B_k^{\epsilon}\left\{B_i^{\xi}, B_j^{\eta}\right\} \otimes I - B_k^{\epsilon} \otimes \left\{B_i^{\xi}, B_j^{\eta}\right\} - \left\{B_i^{\xi}, B_j^{\eta}\right\} \otimes B_k^{\epsilon} - I \otimes B_k^{\epsilon}\left\{B_i^{\xi}, B_j^{\eta}\right\} \\
&\quad \times \left[\left\{B_i^{\xi}, B_j^{\eta}\right\}, B_k^{\epsilon}\right] \otimes I + I \otimes \left[\left\{B_i^{\xi}, B_j^{\eta}\right\}, B_k^{\epsilon}\right] \\
&= \left((\epsilon - \eta)\delta_{jk}B_i^{\xi} + (\epsilon - \xi)\delta_{ik}B_j^{\eta}\right)\right) \otimes I + I \otimes \left((\epsilon - \eta)\delta_{jk}B_i^{\xi} + (\epsilon - \xi)\delta_{ik}B_j^{\eta}\right)\right) \\
&= (\epsilon - \eta)\delta_{jk}\underline{\Delta}^T\left(X_i^{\xi}\right) + x(\epsilon - \xi)\delta_{ik}\underline{\Delta}^T\left(X_j^{\eta}\right) = \underline{\Delta}^T\left((\epsilon - \eta)\delta_{jk}X_i^{\xi} - (\epsilon - \xi)\delta_{ik}X_j^{\eta}\right)\right)
\end{aligned}$$

which finally completes the proof.

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